

# CONVERGENCE ANALYSIS OF STRANG SPLITTING FOR VLASOV-TYPE EQUATIONS

LUKAS EINKEMMER\* AND ALEXANDER OSTERMANN\*

**Abstract.** A rigorous convergence analysis of the Strang splitting algorithm for Vlasov-type equations in the setting of abstract evolution equations is provided. It is shown that under suitable assumptions the convergence is of second order in the time step  $h$ . As an example, it is verified that the Vlasov–Poisson equation in 1+1 dimensions fits into the framework of this analysis. Also, numerical experiments for the latter case are presented.

**Key words.** Strang splitting, abstract evolution equations, convergence analysis, Vlasov–Poisson equations, Vlasov-type equations

**AMS subject classifications.** 65M12, 82D10, 65L05

**1. Introduction.** The most fundamental theoretical description of a (collision-less) plasma comes from the kinetic equation. This so called Vlasov equation is given by (see e.g. [1])

$$\partial_t f(t, \mathbf{x}, \mathbf{v}) + \mathbf{v} \cdot \nabla f(t, \mathbf{x}, \mathbf{v}) + \mathbf{F} \cdot \nabla_{\mathbf{v}} f(t, \mathbf{x}, \mathbf{v}) = 0,$$

where  $\mathbf{x}$  denotes the position and  $\mathbf{v}$  the velocity. The function  $f$  describes a particle-probability distribution in the  $3+3$  dimensional phase space. Since a plasma interacts with the electromagnetic field in a non-trivial manner, the Vlasov equation needs to be coupled to the electromagnetic field.

Depending on the application, either the full Vlasov–Maxwell equations or a simplified model is appropriate. Such models include, for example, the Vlasov–Poisson and the gyrokinetic equations.

Due to the high dimensionality of the equations the most common numerical approach are so called particle methods. In this class of methods, the phase space is left to be continuous and a (large) number of particles with various starting points are advanced in time. This is possible due to the structure of the equations, which implies that a single particle evolves along a trajectory given by an ordinary differential equation. A number of such methods have been developed, most notably the particle-in-cell (PIC) method. Such methods have been extensively used for various applications (see e.g. [7]). The PIC scheme gives reasonable results in case where the tail of the distribution is negligible. If this is not the case the method suffers from numerical noise that only decreases as  $1/\sqrt{n}$ , where  $n$  denotes the number of particles (see e.g. [12] or [8]). Motivated by the considerations above, a number of schemes employing discretization in phase space have been proposed. A comparison of various such methods can be found in [8].

Using a time splitting scheme for the Vlasov–Poisson equations was first proposed by [5] in 1976. In [16] the method was extended to the Vlasov–Maxwell equations. In both cases, second order Strang splitting (see e.g. [14]) is used to advance the solution of the Vlasov equation in time.

Quite a few convergence results are available for semi-Lagrangian methods that employ Strang splitting. For example, in [2], [3] and [17] convergence is shown in the

---

\*Department of Mathematics, University of Innsbruck, Technikerstraße 13, Innsbruck, Austria (lukas.einkemmer@uibk.ac.at, alexander.ostermann@uibk.ac.at). The first author was supported by a scholarship of the Vizerektorat für Forschung, University of Innsbruck.

case of the 1+1 dimensional Vlasov–Poisson equations. Furthermore, the convergence of a special case of the one-dimensional Vlasov–Maxwell equation in the laser-plasma interaction context is investigated in [4].

In this paper, we will consider a class of Vlasov-type equations as abstract evolution equation (i.e. without discretization in space). In this context we will derive sufficient conditions such that the Strang splitting algorithm is convergent of order 2. We will then verify these conditions for the example of the Vlasov–Poisson equations in 1+1 dimensions and present some numerical results.

**2. Setting.** We will investigate the following (abstract) initial value problem

$$\begin{cases} \partial_t f(t) = (A + B)f(t) \\ f(0) = f_0. \end{cases} \quad (2.1)$$

We assume that  $A$  is an (unbounded) linear operator and that  $B$  can be written in the form  $Bf = B(f)f$ , where  $B(f)$  is an (unbounded) linear operator. We will consider this abstract initial value problem on a finite time interval  $[0, T]$ .

Problem (2.1) comprises the Vlasov–Poisson and the Vlasov–Maxwell equations for  $A = -\mathbf{v} \cdot \nabla$  and appropriately chosen  $B$  as special cases. It is also general enough to accommodate the gyrokinetic equations (as stated, for example, in [10]). The Vlasov–Poisson equations are considered in more detail in section 4.

**2.1. The Strang splitting algorithm.** Let  $f_k \approx f(t_k)$  denote the numerical approximation to the solution of (2.1) at time  $t_k = kh$  with step size  $h$ . We assume that the differential equations  $\partial_t f = Af$  and  $\partial_t g = B_{k+1/2}g$ , where  $B_{k+1/2}$  is a first-order approximation to the operator  $B(f(t_k + \frac{h}{2}))$ , can be solved quite efficiently. That this is indeed the case for the Vlasov–Poisson equations is investigated in more detail in section 5.

The idea of Strang splitting is to advance the numerical solution by the recursion  $f_{k+1} = S_k f_k$ , where the (nonlinear) splitting operator  $S_k$  is given by

$$S_k = e^{\frac{h}{2}A} e^{hB_{k+1/2}} e^{\frac{h}{2}A}. \quad (2.2)$$

The precise conditions on  $B_{k+1/2}$  are given in section 3 below. Resolving this recursion, we can compute an approximation to the exact solution at time  $T$  by

$$f_n = \left( \prod_{k=0}^{n-1} S_k \right) f_0 = S_{n-1} \cdots S_0 f_0, \quad (2.3)$$

where  $n$  is an integer chosen together with the step size  $h$  such that  $T = nh$ .

**2.2. Preliminaries.** For the convenience of the reader we collect some well known results that are used quite extensively in section 3.

To bound the remainder term  $R_k(f)$  of a Taylor expansion

$$f(h) = f(0) + hf'(0) + \dots + \frac{h^{k-1}}{(k-1)!} f^{(k-1)}(0) + R_k(f),$$

we will use the integral form

$$R_k(f) = \frac{1}{(k-1)!} \int_0^1 f^{(k)}(hs) (1-s)^{k-1} ds,$$

where  $k \geq 1$ . Note that it is implicitly understood that  $R_k$  is a function of  $h$  as well. However, since we will work mostly with a fixed  $h$ , it is convenient to drop it in the notation of  $R_k$ . For convenience we also define

$$R_0(f) = f.$$

For (unbounded) linear operators it is more convenient to work with the  $\varphi$  functions instead of the remainder term given above.

DEFINITION 2.1 ( $\varphi$  functions). *Suppose that the linear operator  $E$  generates a  $\mathcal{C}_0$  semigroup. Then we define*

$$\begin{aligned} \varphi_0(hE) &= e^{hE}, \\ \varphi_k(hE) &= \int_0^1 e^{(1-\tau)hE} \frac{\tau^{k-1}}{(k-1)!} d\tau, \quad \text{for } k \geq 1. \end{aligned} \tag{2.4}$$

Since we are merely interested in bounds of such functions, we will never directly employ the definition given. Instead we will work exclusively with the following recurrence relation.

LEMMA 2.2. *The  $\varphi$  functions satisfy the recurrence relation*

$$\varphi_k(hE) = \frac{1}{k!} + hE\varphi_{k+1}(hE)$$

and in particular (for  $l \in \mathbb{N}$ )

$$e^{hE} = \sum_{k=0}^{l-1} \frac{h^k}{k!} E^k + h^l E^l \varphi_l(hE).$$

*Proof.* The first relation follows from integration by parts applied to (2.4). The second by using  $\varphi_0 = e^{(\cdot)}$  and applying the first relation repeatedly.  $\square$

The  $\varphi$  functions are used to expand the exponential of some linear operator. In the sense of the previous Lemma, these functions play the same role for an exponential of a linear operator as does the remainder term in Taylor's theorem.

Suppose that the differential equation  $f' = G(f)$  has (for a given initial value) a unique solution. In this case we denote the solution at time  $t$  with initial value  $f(t_0) = f_0$  by the evolution operator  $E_G(t - t_0, f_0)$ .

The Gröbner–Aleksseev formula (also called the nonlinear variation of constants formula) will be employed quite heavily.

THEOREM 2.3 (Gröbner–Aleksseev formula). *Suppose that there exists a unique  $f$  satisfying*

$$\begin{cases} f'(t) = G(f(t)) + R(f(t)) \\ f(0) = f_0 \end{cases}$$

and that  $f' = G(f)$  has (for a given initial value) a unique solution. Then it holds that

$$f(t) = E_G(t, f_0) + \int_0^t \partial_2 E_G(t - \tau, f(\tau)) R(f(\tau)) d\tau.$$

*Proof.* For linear (and possibly unbounded)  $G$ , this formula is proved in [13] by the fundamental theorem of calculus. Here, we prove the extension to nonlinear  $G$ . Let us assume that  $u(t)$  is a solution of  $u'(t) = G(u(t))$ . By differentiating

$$E_G(t - \tau, E_G(\tau - t, u(t))) = u(t)$$

with respect to  $\tau$  we get

$$-\partial_1 E_G(t - \tau, E_G(\tau - t, u(t))) + \partial_2 E_G(t - \tau, E_G(\tau - t, u(t))) G(E_G(\tau - t, u(t))) = 0,$$

which can be rewritten as

$$-\partial_1 E_G(t - \tau, u(\tau)) + \partial_2 E_G(t - \tau, u(\tau)) G(u(\tau)) = 0.$$

The initial value of  $u$  is now chosen such that  $u(\tau) = f(\tau)$  which implies

$$-\partial_1 E_G(t - \tau, f(\tau)) + \partial_2 E_G(t - \tau, f(\tau)) G(f(\tau)) = 0.$$

Altogether we have for  $\varphi(\tau) = E_G(t - \tau, f(\tau))$  (by the fundamental theorem of calculus)

$$\begin{aligned} f(t) - E_G(t, f_0) &= \int_0^t \varphi'(\tau) d\tau \\ &= \int_0^t \left( -\partial_1 E_G(t - \tau, f(\tau)) + \partial_2 E_G(t - \tau, f(\tau)) f'(\tau) \right) d\tau \\ &= \int_0^t \partial_2 E_G(t - \tau, f(\tau)) R(f(\tau)) d\tau, \end{aligned}$$

as desired.  $\square$

Since in some expansions anticommutator relations appear quite naturally, we will employ the notation

$$\{E_1, E_2\} = E_1 E_2 + E_2 E_1,$$

for linear operators  $E_1$  and  $E_2$  (on a suitable domain).

In what follows  $C$  will denote a generic constant that may have different values at different occurrences.

**3. Convergence analysis in the abstract setting.** The problem of splitting an evolution equation into two parts, governed by linear and possibly unbounded operators, has already been investigated in some detail. In [11] it is shown that splitting methods with a given classical order retain this order in the stiff case (under suitable regularity assumptions).

An alternative analysis for Strang splitting in the linear case is given in [14]. The approach presented there is more involved, however, it demands less regularity on the solution. The purpose of this section is to extend this analysis to the abstract initial value problem given by (2.1).

**3.1. Convergence.** Let us begin by defining a suitable notion of consistency for our splitting operator.

**DEFINITION 3.1** (Consistency of order  $p$ ). *The Strang splitting algorithm (2.2) is consistent of order  $p$  if*

$$\|f(t_k + h) - S_k f(t_k)\|_X \leq Ch^{p+1},$$

holds uniformly in  $0 \leq t_k = kh \leq T$ .

Note that for algorithm (2.2), the order of consistency is not necessarily  $p = 2$ . The actual order depends on the properties of the involved operators, and order reduction can show up even in the linear case, see [14].

**THEOREM 3.2 (Convergence).** *Suppose that the Strang splitting (2.2) is consistent of order  $p$  and non-expansive, i.e.*

$$\|S_k\|_{X \leftarrow X} \leq 1. \quad (3.1)$$

*Then it is convergent of order  $p$ , i.e.  $\|f_n - f(t_n)\|_X \leq Ch^p$ .*

*Proof.* The proof is quite standard. We rewrite the global error by means of the following telescopic identity

$$\left( \left( \prod_{k=0}^{n-1} S_k \right) f_0 - f(nh) \right) = \sum_{k=0}^{n-1} \left( \prod_{l=k+1}^{n-1} S_l \right) (S_k f(hk) - f(hk + h)).$$

Then from the consistency bound and by assumption (3.1) (which implies stability) we follow

$$\left\| \left( \prod_{k=0}^{n-1} S_k \right) f_0 - f(nh) \right\|_X \leq \sum_{k=0}^{n-1} \| (S_k f(hk) - f(hk + h)) \|_X \leq Ch^2$$

which is the desired bound.  $\square$

**3.2. Consistency.** It is the purpose of this section to formulate assumptions under which the consistency bound holds for the abstract initial value problem (2.1). To make the derivations less tedious we will adhere to the notation laid out in Remark 3.3.

**REMARK 3.3.** *In this section we will denote the solution of (2.1) at a fixed time  $t_k$  by  $f_0$ . The notation  $f(\tau)$  is then understood to mean  $f(t_k + \tau)$ . The function  $f_0$  is a (possible) initial value for a single time step (i.e., a single application of the splitting operator  $S_k$ ). It is not, in general, the initial value of the solution to the abstract initial value problem as in the previous sections. If we assert that a bound holds uniformly in  $t_k$ , it is implied that it holds for all  $f_0$  in the sense defined here (remember that  $t_k \in [0, T]$ ). Since  $t_k$  is fixed we will use the notation  $B_{1/2}$  and  $S$  instead of  $B_{k+1/2}$  and  $S_k$  respectively.*

Let us start with expanding the exact solution by using the Gröbner–Aleksseev formula (this has been proposed in the context of the nonlinear Schrödinger equation in [15]). It is of emphasize that we consider the linear operator  $A$  as a perturbation of the differential equation given by the non-linear operator  $B$ . This choice is essential for the treatment given here, since it allows us to apply the expansion sequentially without any additional difficulties.

**LEMMA 3.4 (Expansion of the exact solution).** *The exact solution of (2.1) has the formal expansion*

$$\begin{aligned} f(h) &= E_B(h, f_0) + \int_0^h \partial_2 E_B(h - \tau, f(\tau)) A E_B(\tau, f_0) d\tau \\ &+ \int_0^h \int_0^\tau \partial_2 E_B(h - \tau, f(\tau)) A \partial_2 E_B(\tau - \sigma, f(\sigma)) A E_B(\sigma, f_0) d\sigma d\tau \\ &+ \int_0^h \int_0^{\tau_1} \int_0^{\tau_2} \left( \prod_{k=0}^2 \partial_2 E_B(\tau_k - \tau_{k+1}, f(\tau_{k+1})) A \right) f(\tau_3) d\tau_3 d\tau_2 d\tau_1, \end{aligned}$$

where  $\tau_0 = h$ .

*Proof.* Apply the Gröbner–Aleksiev formula three times to equation (2.1).  $\square$

Next we expand the splitting operator  $S$  in a form that is suitable for comparison with the exact solution.

LEMMA 3.5 (Expansion of the splitting operator). *The splitting operator  $S$  has the formal expansion*

$$Sf_0 = e^{hB_{1/2}}f_0 + \frac{h}{2}\{A, e^{hB_{1/2}}\}f_0 + \frac{h^2}{8}\{A, \{A, e^{hB_{1/2}}\}\}f_0 + R_3f_0,$$

where

$$R_3 = \frac{h^3}{16} \int_0^1 (1-s)^2 \left\{ A, \left\{ A, \left\{ A, e^{\frac{hs}{2}A} e^{hB_{1/2}} e^{\frac{hs}{2}A} \right\} \right\} \right\} ds.$$

*Proof.* Let us define the function  $g(s) = e^{\frac{1}{2}sA} e^{hB_{1/2}} e^{\frac{1}{2}sA}$ . The first three derivatives of  $g$  are given by

$$\begin{aligned} g'(s) &= \frac{1}{2} \{A, g(s)\}, \\ g''(s) &= \frac{1}{4} \{A, \{A, g(s)\}\}, \\ g^{(3)}(s) &= \frac{1}{8} \{A, \{A, \{A, g(s)\}\}\}. \end{aligned}$$

From the observation that  $S = g(h)$  and by Taylor's theorem we obtain the result.  $\square$

In the remainder of this section, we use the notation

$$R_i(B)g = h^{-i}(B - B_{1/2})g, \quad i \in \{0, 1, 2\}.$$

THEOREM 3.6 (Consistency). *Suppose that the estimates*

$$\|A^i e^{(h-s)B_{1/2}} R_{2-i}(B) E_B(s, f_0)\|_X \leq C, \quad i \in \{0, 1, 2\} \quad (3.2)$$

$$\|(\partial_2 R_1(B))f_0\|_X \leq C, \quad (3.3)$$

$$\|A^{\delta_{i2}} B_{h/2}^{1+\delta_{i0}} \varphi_{1+\delta_{i0}}(hB_{1/2}) A^{1+\delta_{i1}} f_0\|_X \leq C, \quad i \in \{0, 1, 2\} \quad (3.4)$$

$$\|A^{\delta_{i2}} R_{1+\delta_{i0}}(\partial_2 E_B(\cdot, f_0)) A^{1+\delta_{i1}} f_0\|_X \leq C, \quad i \in \{0, 1, 2\} \quad (3.5)$$

hold uniformly in  $t$  and in  $s \in [0, h]$ . In addition, suppose that for all  $k_j \in \mathbb{N}$ , with  $\sum_{j=1}^{i+1} k_j = 3 - i$ , the estimates

$$\left\| \left( \prod_{j=1}^i D_j^{k_j} \partial_2 E_B(s_j, f(\sigma_j)) A \right) \partial_{s_{i+1}}^{k_{i+1}} E_B(s_{i+1}, f_0) \right\|_X \leq C, \quad i \in \{1, 2\} \quad (3.6)$$

$$\left\| \left( \prod_{k=1}^3 \partial_2 E_B(s_k - \sigma_k, f(\sigma_k)) A \right) f(s) \right\|_X \leq C, \quad (3.7)$$

$$\| \{A, \{A, \{A, e^{\frac{s}{2}A} e^{hB_{1/2}} e^{\frac{s}{2}A}\}\}\} f_0 \|_X \leq C, \quad (3.8)$$

hold uniformly in  $t$  as well as in  $s \in [0, h]$ ,  $s_j \in [0, h]$ , and  $\sigma_j \in [0, h]$ , where  $D_j^{k_j}$  is a differential operator of order  $k_j$  in the variables  $s_j$  and  $\sigma_j$ . Then the Strang splitting (2.2) is consistent of order 2.

*Proof.* We have to compare terms of order 0, 1, and 2 in Lemma 3.4 and Lemma 3.5 and show that the remaining terms of order 3 can be bounded as well.

*Terms of order 0.* We can rewrite  $E_B(\tau, f_0)$  as  $u(\tau)$ , where  $u(\tau)$  satisfies the differential equation

$$u' = B_{1/2}u + (B - B_{1/2})u,$$

with initial value  $f_0$ . Employing the variation of constants formula we get

$$u(h) = e^{hB_{1/2}}f_0 + \int_0^h e^{(h-\tau)B_{1/2}}(B - B_{1/2})E_B(\tau, f_0) d\tau.$$

Assumption (3.2) with  $i = 0$  gives the desired order.

*Terms of order 1.* For

$$g(\tau) = e^{(h-\tau)B_{1/2}}Ae^{\tau B_{1/2}}f_0, \quad k(\tau) = \partial_2 E_B(h - \tau, f(\tau))AE_B(\tau, f_0)$$

we get (by use of the trapezoidal rule)

$$\frac{h}{2} [g(0) + g(h)] - \int_0^h k(\tau) d\tau = \frac{h}{2} [g(0) - k(0) + g(h) - k(h)] - \frac{h^3}{2} \int_0^1 \theta(1-\theta)k''(\theta h) d\theta.$$

First, let us compare  $g(h)$  and  $k(h)$

$$g(h) - k(h) = A [e^{hB_{1/2}}f_0 - E_B(h, f_0)],$$

which is the same as before, except that we have an additional  $A$  to the left of the expression. We are left with assumption (3.2) with  $i = 1$ . Second, we have to compare  $g(0)$  and  $k(0)$

$$g(0) - k(0) = \partial_2 [e^{hB_{1/2}} - E_B(h, f_0)] Af_0,$$

where we can pull out  $\partial_2$  as it does not effect the linear operator  $e^{hB_{1/2}}$ . To handle the derivative with respect to the initial value, we expand both terms (using the  $\varphi$  functions for the exponential) and get

$$g(0) - k(0) = h^2 \left[ (\partial_2 R_1(B))f_0 + B_{1/2}^2 \varphi_2(hB_{1/2}) - R_2(\partial_2 E_B(\cdot, f_0)) \right] Af_0.$$

The first term is bounded by assumption (3.3). The second term is bounded by assumption (3.4) with  $i = 0$  and the third term by assumption (3.5) with  $i = 0$ .

Finally, we have to estimate the remainder term of the quadrature rule which is bounded by assumption (3.6) with  $i = 1$ .

*Terms of order 2.* For the functions  $g(\tau, \sigma) = e^{(h-\tau)B_{1/2}}Ae^{(\tau-\sigma)B_{1/2}}Ae^{\sigma B_{1/2}}f_0$  and  $k(\tau, \sigma) = \partial_2 E_B(h - \tau, f(\tau))A\partial_2 E_B(\tau - \sigma, f(\sigma))AE_B(\sigma, f_0)$  we employ a quadrature rule

$$\begin{aligned} & \frac{h^2}{8} [g(0, 0) + 2g(h, 0) + g(h, h)] - \int_0^h \int_0^\tau k(\tau, \sigma) d\sigma d\tau \\ &= \frac{h^2}{8} [g(0, 0) + 2g(h, 0) + g(h, h) - k(0, 0) - 2k(h, 0) - k(h, h)] + d, \end{aligned}$$

where  $d$  is the remainder term. From this, it can be shown that we have to bound the following three terms

$$\begin{aligned} & A^2 [e^{hB_{1/2}} f_0 - E_B(h, f_0)] \\ & [e^{hB_{1/2}} - \partial_2 E_B(h, f_0)] A^2 f_0 = h [B_{1/2} \varphi_1(hB_{1/2}) - R_1(\partial_2 E_B(\cdot, f_0))] A^2 f_0 \\ & 2A [e^{hB_{1/2}} - \partial_2 E_B(h, f_0)] A f_0 = 2hA [B_{1/2} \varphi_1(hB_{1/2}) - R_1(\partial_2 E_B(\cdot, f_0))] A f_0. \end{aligned}$$

The first term we can bound by using assumption (3.2) with  $i = 2$ . In addition, we can bound the second and third term using assumption (3.4) with  $i = 1$  and  $i = 2$  and assumption 3.5 with  $i = 1$  and  $i = 2$  respectively. Finally, the remainder term depends on the partial derivatives of  $k(\tau, \sigma)$  and can be bound by (3.6) with  $i = 2$ .

*Terms of order 3.* In order to bound the remainder terms in the expansion of the exact solution as well as the approximate solution, we need assumption (3.7) and (3.8) respectively.  $\square$

**4. Convergence analysis for the Vlasov–Poisson equations.** We will consider the Vlasov–Poisson equations in 1+1 dimensions, i.e.

$$\begin{cases} \partial_t f(t, x, v) = -v \partial_x f(t, x, v) - E(f(t, \cdot, \cdot), x) \partial_v f(t, x, v) \\ \partial_x E(f(t, \cdot, \cdot), x) = \int_{\mathbb{R}} f(t, x, v) dv - 1 \\ f(0, x, v) = f_0(x, v), \end{cases}$$

with periodic boundary conditions in space. The domain of interest has length  $L$ . Thus, for all  $x \in \mathbb{R}$

$$f(t, x, v) = f(t, x + L, v).$$

As will be apparent in the next section it is unnecessary to impose boundary conditions in the velocity direction. This is due to the fact that for a function  $f_0$  with compact support in the velocity direction the solution will continue to have compact support for all finite time intervals  $[0, T]$  (see Theorem 4.1).

**4.1. Definitions and notation.** The purpose of this section is to introduce the notations and mathematical spaces necessary for giving existence, uniqueness, and regularity results as well as to conduct the estimates necessary for showing consistency and stability.

For estimation we will use the Banach space  $L^1([0, L] \times \mathbb{R})$  exclusively. This is reasonable as the solution  $f$  represents a probability density function. As such the  $L^1$  norm is conserved for the exact (as well as the approximate) solution. Nevertheless, all the estimations could be done as well, for example, in  $L^\infty([0, L] \times \mathbb{R})$ .

However, we need some regularity of the solution. This can be seen from the assumptions of Theorem 3.6, where we have to apply a number of differential operators to the solution  $f(t)$ . Thus, we introduce the following spaces of continuously differentiable functions

$$\begin{aligned} \mathcal{C}_{\text{per},c}^m &= \{f \in \mathcal{C}^m(\mathbb{R}^2, \mathbb{R}); \forall x, v: (f(x + L, v) = f(x, v)) \wedge (\text{supp} f(x, \cdot) \text{ compact})\}, \\ \mathcal{C}_{\text{per}}^m &= \{f \in \mathcal{C}^m(\mathbb{R}, \mathbb{R}); \forall x: f(x + L) = f(x)\}. \end{aligned}$$

Together with the norm of uniform convergence of all derivatives up to order  $m$ , the spaces  $\mathcal{C}_{\text{per},c}^m$  and  $\mathcal{C}_{\text{per}}^m$  are turned into Banach spaces.



We also have to consider spaces that involve time. To that end let us define

$$\mathcal{C}^m(0, T; \mathcal{C}^m) = \left\{ f \in \mathcal{C}^m([0, T], \mathcal{C}^0); (f(t) \in \mathcal{C}^m) \wedge \left( \sup_{t \in [0, T]} \|f(t)\|_{\mathcal{C}^m} < \infty \right) \right\},$$

where  $\mathcal{C}^m$  is taken as either  $\mathcal{C}_{\text{per},c}^m$  or  $\mathcal{C}_{\text{per}}^m$ . It should be noted that if it can be shown that the solution  $f$  of the Vlasov–Poisson equations lies in the space  $\mathcal{C}^m(0, T; \mathcal{C}^m)$ , we can bound all derivatives (in space) up to order  $m$  uniformly in  $t \in [0, T]$ .

**4.2. Existence, uniqueness, and regularity.** In this section we recall the existence, uniqueness, and regularity results of the Vlasov–Poisson equations in 1+1 dimensions. The theorem is stated with a slightly different notation in [3] and [2].

**THEOREM 4.1.** *Assume that  $f_0 \in \mathcal{C}_{\text{per},c}^m$  is non-negative, then  $f \in \mathcal{C}^m(0, T; \mathcal{C}_{\text{per},c}^m)$  and  $E(f(t, \cdot, \cdot), x)$  as a function of  $(t, x)$  lies in  $\mathcal{C}^m(0, T; \mathcal{C}_{\text{per}}^m)$ . In addition, we can find a number  $Q(T) > 0$  such that for all  $t \in [0, T]$  and  $x \in \mathbb{R}$  it holds that  $\text{supp} f(t, x, \cdot) \subset [-Q(T), Q(T)]$ .*

*Proof.* A proof can be found in [9].  $\square$

We also need a regularity result for the electric field that does not directly result from a solution of the Vlasov–Poisson equations, but from some generic function  $f$  (e.g., computed from an application of a splitting operator to  $f_0$ ).

**COROLLARY 4.2.** *For  $f \in \mathcal{C}_{\text{per},c}^m$  it holds that  $E(f, \cdot) \in \mathcal{C}_{\text{per}}^m$ .*

*Proof.* The result follows from the proof of Theorem 4.1. In addition, in the 1+1 dimensional case it can also be followed from the exact representation of the electromagnetic field that is given in (5.2) below.  $\square$

It should also be noted that due to the proof of Theorem 4.1, the regularity results given can be extended to the differential equations generated by  $B$  and  $B_{1/2}$ . Thus, Theorem 4.1 remains valid if  $E_B(t, f_0)$  or  $e^{tB_{1/2}} f_0$  is substituted for  $f(t)$ .

**4.3. Consistency.** The most challenging task in proving the assumptions of Theorem 3.6 is to control the derivative of  $E_B$  with respect to the initial value. The following lemma accomplishes that.

**LEMMA 4.3.** *The function*

$$\begin{aligned} \mathcal{C}_{\text{per},c}^m \times \mathcal{C}_{\text{per},c}^\ell &\rightarrow \mathcal{C}_{\text{per},c}^{\min(m-1, \ell)} \\ (u_0, g) &\mapsto \partial_2 E_B(t, u_0)g, \end{aligned}$$

*is well defined.*

*Proof.* We solve  $u'(t) = Bu(t)$  and  $u(0) = u_0$ . Motivated by the method of characteristics we can write

$$\begin{aligned} V_{u_0}'(t) &= E(u(t, \cdot, \cdot), x) \\ V_{u_0}(0) &= v \\ u(t, x, v) &= u_0(x, V_{u_0}(t)(x, v)). \end{aligned}$$

We now compute the Gâteaux derivative with respect to the direction  $g$  and get

$$\partial_h E_B(t, u_0 + hg)(x, v)|_{h=0} = (\partial_2 u_0)(x, V_{u_0}(t)(x, v)) V_g(t)(x, v) + g(x, V_{u_0}(t)(x, v)),$$

since  $V$  is linear with respect to the initial value. From this representation the result follows.  $\square$

In addition, it is useful to have an explicit representation of the Taylor series remainder terms of  $B$  and  $B_{1/2}$ . However, up until now we have only specified that

$B_{1/2}$  is a first order approximation to  $B$ . To perform the analysis that follows we will approximate  $B_{1/2}$  by an evaluation of  $B(\cdot)$ . The details are given in the following definition.

DEFINITION 4.4 (Structure of  $B_{1/2}$ ). *Let us define*

$$B_{1/2}f = B(f_{h/2})f,$$

where  $f_{h/2}$  is an approximation to  $f(h/2)$  of order 1.

It remains, however, to verify that this ansatz produces an approximation of order 1, i.e. it is consistent with the assumptions we made above.

LEMMA 4.5.  $B_{1/2}$  is an approximation of order 1 to  $B$ .

*Proof.* Since  $f(h/2) = f(0) + \frac{h}{2}f'(0) + \mathcal{O}(h^2)$  the assumption implies that  $f_{h/2} = f(0) + \frac{h}{2}f'(0) + \mathcal{O}(h^2)$ . Expanding  $B$  we get

$$B\left(f\left(\frac{h}{2}\right)\right) = B(f(0)) + \frac{h}{2}B'(f(0))f'(0) + \mathcal{O}(h^2).$$

Comparing this with the expansion of  $B_{1/2}$

$$B(f_{h/2}) = B(f(0)) + \frac{h}{2}B'(f(0))f'(0) + \mathcal{O}(h^2)$$

our result follows.  $\square$

The following two lemmas present time derivatives up to order 2 of  $Bf$ ,  $Bf_{h/2}$  and  $E_B(t, f_0)$  which follow from a simple calculation. Let us start with the derivatives of the operator  $B$  and  $B_{1/2}$  applied to some  $f(t) = f(t, \cdot, \cdot)$ .

LEMMA 4.6. *For  $f$  sufficiently often continuously differentiable, we have*

$$\begin{aligned} \partial_t Bf(t, x, v) &= E(f'(t), x) \partial_v f(t, x, v) + E(f(t), x) \partial_v f'(t, x, v) \\ \partial_t^2 Bf(t, x, v) &= E(f''(t), x) \partial_v f(t, x, v) \\ &\quad + 2E(f'(t), x) \partial_v f'(t, x, v) + E(f(t), x) \partial_v f''(t, x, v) \\ \partial_t B_{1/2}f(t, x, v) &= E(f_{h/2}, x) \partial_v f'(t, x, v) \\ \partial_t^2 B_{1/2}f(t, x, v) &= E(f_{h/2}, x) \partial_v f''(t, x, v) \end{aligned}$$

*Proof.* From the relation  $Bf(t, x, v) = E(f(t), x) \partial_v f(t, x, v)$  and  $B_{1/2}f(t, x, v) = E(f_{h/2}, x) \partial_v f(t, x, v)$  the result follows by the product rule.  $\square$

Also we have to compute some derivatives of the evolution operator  $E_B(t, f_0)$  with respect to time.

LEMMA 4.7. *For  $f$  sufficiently often continuously differentiable, we have*

$$\begin{aligned} \partial_t E_B(t, f_0) &= BE_B(t, f_0) \\ \partial_t^2 E_B(t, f_0) &= E(E_B(t, f_0), x) \partial_v (BE_B(t, f_0)) + E(BE_B(t, f_0), x) \partial_v E_B(t, f_0) \\ \partial_t (\partial_2 E_B(t, f_0)) &= E(E_B(t, f_0), x) \partial_v (\partial_2 E_B(t, f_0)) + E(\partial_2 E_B(t, f_0), x) \partial_v E_B(t, f_0) \\ \partial_t^2 (\partial_2 E_B(t, f_0)) &= E(BE_B(t, f_0), x) \partial_v (\partial_2 E_B(t, f_0)) \\ &\quad + E(E_B(t, f_0), x) \partial_v (\partial_t (\partial_2 E_B(t, f_0))) \\ &\quad + E(\partial_t (\partial_2 E_B(t, f_0)), x) \partial_v E_B(t, f_0) \\ &\quad + E(\partial_2 E_B(t, f_0), x) \partial_v BE_B(t, f_0) \end{aligned}$$

*Proof.* From the relation  $Bf(t, x, v) = E(f(t), x)\partial_v f(t, x, v)$  the result follows by a simple calculation.  $\square$

It is also necessary to investigate the behavior of the  $\varphi$  functions introduced in Definition 2.1.

LEMMA 4.8. *For the Vlasov–Poisson equations the functions  $E^{\min\{i,1\}}\varphi_i(hE)$  with  $E \in \{A, B_{1/2}\}$  are maps from  $\mathcal{C}_{\text{per},c}^m$  to  $\mathcal{C}_{\text{per},c}^m$  for all  $i \in \mathbb{N}$ .*

*Proof.* For  $i = 0$  we have

$$e^{-hv\partial_x} f_0(x, v) = f_0(x - vh, v),$$

and

$$e^{-hE(f_{h/2}, x)\partial_v} f_0(x, v) = f_0(x, v - E(f_{h/2}, x)h).$$

This clearly doesn't change the differentiability of  $f_0$ . For the  $\varphi$  functions we proceed by induction. Since by Definition 2.1 we have

$$hE\varphi_{k+1}(hE) = \varphi_k(hE) - \frac{1}{k!}I,$$

from which the desired result follows.  $\square$

Now we are able to show that all the assumptions of Theorem 3.6 are fulfilled and that we thus have consistency of order 2. This is the content of theorem 4.9.

THEOREM 4.9. *Suppose  $f_0 \in \mathcal{C}_{\text{per},c}^3$ ,  $f_0$  is non-negative and  $f_{h/2}$  is an approximation to  $f(h/2)$  of order 1. Then Strang splitting for the Vlasov–Poisson equations is consistent of order 2.*

*Proof.* The proof proceeds by noting that the solution has compact support (for a finite time interval), i.e. we can estimate  $v$  by some constant  $Q$ . On the other hand it is clear that for  $f_0 \in \mathcal{C}_{\text{per},c}^{m+1}$  we get  $Af_0 \in \mathcal{C}_{\text{per},c}^m$  and  $B_{1/2}f_0 \in \mathcal{C}_{\text{per},c}^m$ . The same is true for  $B$  as can be seen by Corollary 4.2. Noting that by Lemma 4.6 terms of the form  $R_i(B)$  are mappings from  $\mathcal{C}_{\text{per},c}^{m+i}$  to  $\mathcal{C}_{\text{per},c}^m$  and that by Lemma 4.8 the  $\varphi$  functions are mappings from  $\mathcal{C}_{\text{per},c}^m \rightarrow \mathcal{C}_{\text{per},c}^m$  we can conclude that after applying all operators in assumptions (3.2), (3.3), (3.4), and (3.5) we get a continuous function. By the regularity results we can bound these functions uniformly in time. The same argument also shows the validity of the bound in assumption (3.8).

Finally, with the help of Lemma 4.3 and 4.7 together with the above observations we can bound assumptions (3.6) and (3.7).  $\square$

**4.4. Convergence.** We are now in the position to prove second order convergence of Strang splitting for the Vlasov–Poisson equations in  $L^1$ . The same result holds literally in  $L^\infty$  (or any other  $L^p$  space).

THEOREM 4.10. *Suppose  $f_0 \in \mathcal{C}_{\text{per},c}^3$ ,  $f_0$  is non-negative and  $f_{h/2}$  is an approximation to  $f(h/2)$  of order 1. Then Strang splitting for the Vlasov–Poisson equations is convergent of order 2.*

*Proof.* By Theorem 4.9 and 3.2 it is sufficient to show that

$$\left\| e^{\frac{h}{2}A} e^{hB_{k+1/2}} e^{\frac{h}{2}A} f \right\|_1 \leq \|f\|_1.$$

It is evident from the proof of Lemma 4.8 that the above operators can be represented as a translation only. Since a translation does not change the  $L^1$  norm our result follows.  $\square$

**5. Numerical experiments.** In this section we present some numerical experiments. Even if we neglect space discretization for the moment, we have to settle on some  $f_{h/2}$  that is an approximation of  $f(h/2)$  of order 1. This can be achieved by Taylor series expansion, interpolation of previously computed values, or by making an additional Lie–Trotter time step of length  $h/2$ . Since we are interested in time integration only, we choose the latter. This method is trivial to implement (once the Strang splitting scheme is implemented) and doesn't suffer from the numeric differentiation problems of a Taylor expansion. Thus, one possible choice is to use

$$f_{h/2} = e^{\frac{h}{2}B(f_0)}e^{\frac{h}{2}A}f_0 \quad (5.1)$$

in our simulations. However, since the semigroup generated by  $B(f_0)$  can be represented as a translation in velocity (see the proof of Lemma 4.8) and the electric field depends only on the average of the density function with respect to velocity (i.e. it depends only on the charge density), it is possible to drop the first factor in (5.1) without affecting the resulting electric field. Since the computation of the second operator in (5.1) is the first step in the Strang splitting algorithm, this leads to a computationally efficient scheme. This scheme is also employed in [16], for example. However, no argument why second order accuracy is retained is given there.

To compute the electric field we will use the following formula (see e.g. [3])

$$E(f(t, \cdot, \cdot), x) = \int_0^L K(x, y) \left( \int_{\mathbb{R}} f(t, y, v) dv - 1 \right) dy, \quad (5.2)$$

$$K(x, y) = \begin{cases} \frac{y}{L} - 1 & 0 \leq x < y, \\ \frac{y}{L} & y < x \leq L. \end{cases}$$

For space discretization we will employ the discontinuous Galerkin method described in [16]. The approximation is of order 2 with 80 cells in both the space and velocity direction. In [16] the coefficients up to order 2 are given. However, it is no difficult to use a computer program to compute the coefficients to arbitrary order.

**5.1. Landau damping.** The Vlasov–Poisson equations in 1+1 dimensions together with the initial value

$$f_0(x, v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} (1 + \alpha \cos(0.5x)),$$

is called Landau damping. For  $\alpha = 0.01$  the problem is called linear or weak Landau damping and for  $\alpha = 0.5$  it is referred to as strong or non-linear Landau damping. As can be seen, for example, in [8, 6] and [18] Landau damping is a popular test problem for Vlasov codes.

For comparison we display the error of the Strang splitting algorithm together with the error for Lie–Trotter splitting (a method that is expected to be of order 1). Since we are mainly interested in the time integration error and there is no analytical solution of the problem available, we compare the error for different step sizes with a reference solution computed with  $h = 3.9 \cdot 10^{-3}$ . The error is computed in the discrete  $L^1$  norm at time  $t = 1$ . The results are given in Figure 5.1.

**6. Conclusion.** In this paper sufficient conditions are given that guarantee convergence of order 2 for the Strang splitting algorithm in the case of Vlasov–type equations. It is then shown that the Vlasov–Poisson equations in 1+1 dimensions

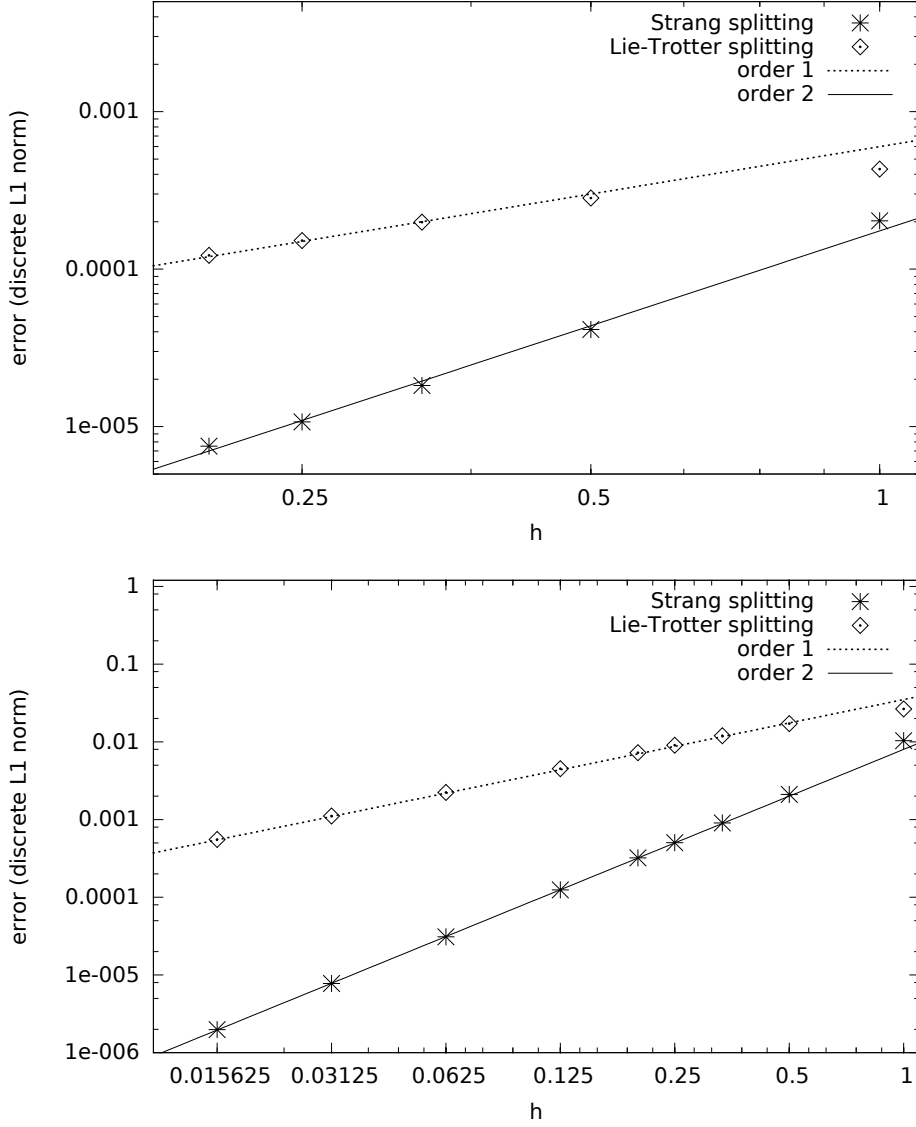


FIG. 5.1. Error of the particle density function  $f(1, \cdot, \cdot)$  for Strang and Lie-Trotter splitting respectively, where  $\alpha = 0.01$  (top) and  $\alpha = 0.5$  (bottom).

is an example of a Vlasov-type equation, i.e. that it fits into the framework of the analysis conducted. For simulation on a computer, however, an approximation has to be made (i.e. some sort of space discretization has to be introduced). This approximation is not included in the analysis done here. Nevertheless, the numerical experiments suggest that second order convergence is retained in the fully discretized case as well.

## REFERENCES

- [1] E.A. BELLI, *Studies of numerical algorithms for gyrokinetics and the effects of shaping on plasma turbulence*, PhD thesis, Princeton University, 2006.
- [2] N. BESSE, *Convergence of a semi-Lagrangian scheme for the one-dimensional Vlasov-Poisson system*, SIAM J. Numer. Anal., 42 (2005), pp. 350–382.
- [3] ———, *Convergence of a high-order semi-Lagrangian scheme with propagation of gradients for the one-dimensional Vlasov-Poisson system*, SIAM J. Numer. Anal., 46 (2008), pp. 639–670.
- [4] M. BOSTAN AND N. CROUSEILLES, *Convergence of a semi-Lagrangian scheme for the reduced Vlasov-Maxwell system for laser-plasma interaction*, Numer. Math., 112 (2009), pp. 169–195.
- [5] C.Z. CHENG AND G. KNORR, *The integration of the Vlasov equation in configuration space*, J. Comput. Phys., 22 (1976), pp. 330–351.
- [6] N. CROUSEILLES, E. FAOU, AND M. MEHRENBARGER, *High order Runge-Kutta-Nyström splitting methods for the Vlasov-Poisson equation*.  
<http://hal.inria.fr/inria-00633934/PDF/cfm.pdf>.
- [7] M.R. FAHEY AND J. CANDY, *GYRO: A 5-d gyrokinetic-Maxwell solver*, Proceedings of the ACM/IEEE SC2004 Conference, (2008), p. 26.
- [8] F. FILBET AND E. SONNENDRÜCKER, *Comparison of Eulerian Vlasov solvers*, Computer Physics Communications, 150 (2003), pp. 247–266.
- [9] R. T. GLASSEY, *The Cauchy Problem in Kinetic Theory*, SIAM, 1996.
- [10] T.S. HAHM, L. WANG, AND J. MADSEN, *Fully electromagnetic nonlinear gyrokinetic equations for tokamak edge turbulence*, Physics of Plasmas, 16 (2009), p. 022305.
- [11] E. HANSEN AND A. OSTERMANN, *Dimension splitting for evolution equations*, Numer. Math., 108 (2008), pp. 557–570.
- [12] R.E. HEATH, I.M. GAMBA, P.J. MORRISON, AND C. MICHLER, *A discontinuous Galerkin method for the Vlasov-Poisson system*, J. Comput. Phys., 231 (2012), pp. 1140–1174.
- [13] H. HOLDEN, C. LUBICH, AND N.H. RISEBRO, *Operator splitting for partial differential equations with Burgers nonlinearity*. To appear in Math. Comp.
- [14] T. JAHNKE AND C. LUBICH, *Error bounds for exponential operator splittings*, BIT, 40 (2000), pp. 735–744.
- [15] C. LUBICH, *On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger equations*, Math. Comput., 77 (2008), pp. 2141–2153.
- [16] A. MANGENEY, F. CALIFANO, C. CAVAZZONI, AND P. TRAVNICEK, *A numerical scheme for the integration of the Vlasov-Maxwell system of equations*, J. Comput. Phys., 179 (2002), pp. 495–538.
- [17] T. RESPAUD AND E. SONNENDRÜCKER, *Analysis of a new class of forward semi-Lagrangian schemes for the 1D Vlasov Poisson equations*, Numer. Math., 118 (2011), pp. 329–366.
- [18] J.A. ROSSMANITH AND D.C. SEAL, *A positivity-preserving high-order semi-Lagrangian discontinuous Galerkin scheme for the Vlasov-Poisson equations*, J. Comput. Phys., 230 (2011), pp. 6203–6232.